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Perturbation Theory with Higher Derivative Couplings

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The formalism of partial differential equations with respect to coupling constants is used to develop a covariant perturbation theory for the interpolating fields and the S matrix when the coupling terms in the Larangian density involve arbitrary (first and higher) derivatives. Through the notion of pure noncovariant contractions, the free-field T and the (covariant) T^* products can be related to each other, allowing us to avoid the Hamiltonian density altogether when dealing with the S matrix. The important ingredients in our approach are (1) the adiabatic switching on and off of the interactions in the infinite past and future, respectively, and (2) the vanishing of four-dimensional delta functions and their derivatives at zero space-time points. The latter ingredient is a prerequisite that our formalism and the canonical formalism be consistent with each other, and on the other hand, it is supported by the dimensional regularization. Corresponding to any Lagrangian, the generalized interaction Hamiltonian density is defined from the covariant S matrix with the help of the pure noncovariant contractions. This interaction Hamiltonian density reduces to the usual one when the Lagrangian density depends on just first derivatives and when the usual canonical formalism can be applied.

1. INTRODUCTION

Theories with higher derivatives (second and higher) in the Lagrangian density can occur quite naturally in various areas of physics (Bernard and Duncan, 1975; Simon, 1990; Barua and Gupta, 1977). For example, in the treatment of higher spin fields with the use of ghost fields, the higher derivative couplings are unavoidable (Barua and Gupta, 1977). Another example of a theory with higher derivatives occurs in general relativity where quantum corrections contain higher derivatives of the metric (Birrel and Davis, 1982), or where nonlinear sigma models of string theory predict terms of order R^2 or higher (Bernard and Duncan, 1975; Simon, 1990; de Alwis, 1986). Of course, the classical case of a theory with higher derivatives is Dirac's (1938) relativistic model of the radiating electron.

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The canonical formalism for fields with higher derivatives in the free part of the Lagrangian density has been discussed rather extensively (Bernard and Duncan, 1975; Simon, 1990). In this article we deal with the situation where the free part of the Lagrangian density involves only first derivatives and the coupling terms contain higher derivatives. Now, from the interaction Lagrangian with higher derivatives one can remove second and higher time derivatives by carrying out either covariant or noncovariant field transformations (Barua and Gupta, 1977), so that the transformed Lagrangian contains only higher space derivatives which (with suitable definitions of canonical momenta) yields the Hamiltonian density. This procedure, however, is very model dependent and, as a rule, yields a rather complicated Hamiltonian density, which in turn yields a rather complicated expression for the scattering (S) matrix. It is clear that in order to treat a general case of higher derivative couplings, one has to formulate a theory that goes beyond the canonical formalism. We believe that such a theory can be formulated through the formalism of partial differential equations with respect to coupling constants (PDECC) (Šoln, 1972, 1978), which has been used in the formulation of the covariant perturbation theory for chiral Lagrangians (Šoln, 1973). However, unlike earlier work (Šoln, 1972, 1973, 1978), here the formulation of the covariant PDECC formalism will be done without the Hamiltonian density.

A rather compelling reason for developing the S-matrix formulation in which the knowledge of the Hamiltonian density is not necessary is the difficulty in showing the Lorentz invariance itself. Namely, when tackling the question of the Lorentz invariance of the S matrix within the canonical formalism for a Lagrangian density with higher derivatives, one needs to know the corresponding Hamiltonian density, which, however, quite generally, if it can be written at all, is a very complicated noncovariant expression (and seldom can be written in a closed form). Furthermore, when a term in the Lagrangian density contains more derivatives than fields, the Hamiltonian density does not exist until auxiliary fields are introduced, as originally noted by Ostrogradsky (1850).

The covariant PDECC formalism developed here to treat fields and the S matrix, when the free part of the Lagrangian density involves only first derivatives and the coupling terms contain higher derivatives, is consistent with the so-called Lehman–Symanzik–Zimmermann (LSZ) field theory formulation (Lehman *et al.*, 1955), which assumes that the free fields satisfy the usual differential (Klein–Gordon, Dirac) equations. Also, as in the LSZ formulation, we take that the constant of motion means a field quantity which commutes with the S matrix. The derived covariant S matrix will satisfy automatically the naive version of Matthews' (1949) theorem to all orders in a perturbation theory. This should be contrasted with the fact that within the canonical formalism, which relays on the S matrix given as a T product in terms of the interaction Hamiltonian density, one can only conjecture but not actually prove (even for Lagrangians with no more than one derivative acting on each field) that the naive version of Matthews' theorem is correct in a perturbation theory (Bernard and Duncan, 1975). Here, the naive version of Matthews' theorem means: The Feynman rules are the ones obtained by using the interaction Lagrangian to determine the vertices and the covariant T^* product (Nishijima, 1969) to determine the propagators (Bernard and Duncan, 1975).

The main reason we can avoid the Hamiltonian altogether within our covariant PDECC formalism is because by introducing the pure noncovariant T_n product (with the corresponding pure noncovariant contractions) we can transform directly a T product into a covariant T^* product (and vice versa). The adiabatic switching on and off of interactions in the infinite past and future, respectively, is explicitly assumed in our covariant formalism; this allows us to verify that the derived covariant S matrix indeed connects properly the asymptotic field quantities at the infinite past and the future. The vanishing of the four-dimensional delta function (Bernard and Duncan, 1975; Barua and Gupta, 1977; Capper and Liebbrandt, 1973, 1974; Tataru, 1975; 't Hooft and Veltman, 1972) and its derivatives at zero space-time points is yet another important ingredient in the covariant PDECC formalism. This ingredient is actually a physical necessity, as otherwise the canonical formalism (for Lagrangian densities with first derivatives) and the covariant PDECC formalism would not be consistent with each other. Of course, on the formal level, the vanishing of the four-dimensional delta function and its derivatives at zero arguments can be justified with the dimensional regularization ('t Hooft and Veltman, 1972).

Although in the covariant PDECC formalism one does not need the Hamiltonian density, nevertheless, with the help of pure noncovariant contractions from the explicitly covariant S matrix, we can define the generalized interaction Hamiltonian density for an arbitrary Lagrangian density. For a Lagrangian density with first derivatives, this interaction Hamiltonian density reduces to the one that is derived from the canonical formalism.

In Section 2 the algebra of time-ordered products is developed. Here the T and T^* products are connected through (pure noncovariant) T_n products. The covariant version of the PDECC formalism, with the arbitrary derivative coupling terms in the Lagrangian density, for the S matrix and interpolating field quantities is developed in Section 3. Here the explicitly covariant S matrix, the covariant PDECC for interpolating fields, as well as the generalized Hamiltonian density are given. Section 4 is devoted to treating some specific examples with an emphasis on analysis of the derived Hamiltonian densities. The results are discussed in Section 5, where the conclusion is also given. In Appendix A we give pertinent examples of T_n contractions. Some outlines of dimensional regularization are discussed in Appendix B and some PDECC relations for theories with canonical Hamiltonians are explained in Appendix C.

2. ALGEBRA OF TIME-ORDERED PRODUCTS

In the formulation of the covariant PDECC, the algebra of T (covariant) T^*, and (noncovariant) T_n products play very important roles, so this whole section is devoted to it.

According to Šoln (1973) and Nishijima (1950, 1969), a T^* product of free fields is defined as

$$T^*\left(\left(\frac{\partial}{\partial x_{\mu_1}}\frac{\partial}{\partial x_{\mu_2}}\cdots\right)\phi_f(x)\left(\frac{\partial}{\partial y_{\nu_1}}\frac{\partial}{\partial y_{\nu_2}}\cdots\right)\phi_f(y)\cdots\right)$$
$$=\left(\frac{\partial}{\partial x_{\mu_1}}\frac{\partial}{\partial x_{\mu_2}}\cdots\right)\left(\frac{\partial}{\partial y_{\nu_1}}\frac{\partial}{\partial y_{\nu_2}}\cdots\right)\cdots T(\phi_f(x)\phi_f(y)\cdots) \tag{2.1}$$

where ϕ_f (meaning either ϕ_{in} or ϕ_{out}) denotes a set of independent free fields. It is immediately evident from (2.1) that T and T^* contractions between free field ϕ_f operators are different, for while T contractions may not be covariant objects in general, the T^* contractions always are. Let us take a simple case of a T contraction involving scalar fields carrying some internal (isospin, etc.) indices a and b:

$$\partial_\mu\phi_f^a(x)^{\bullet}\,\partial_\nu\phi_f^b(y)^{\bullet}=\langle 0;f|T\frac{\partial}{\partial x_\mu}\phi_f^a(x)\frac{\partial}{\partial y_\nu}\phi_f^b(y)|0;f\rangle$$
$$=[\partial_\mu\phi_f^{a\bullet}(x)\,\partial_\nu\phi_f^{b\bullet}(x)]^*-in_\mu n_\nu\delta_{ab}\delta_4(x-y) \tag{2.2a}$$
$$n_\mu=g_\mu^4$$

where the first term is the T^* contraction,

$$[\partial_\mu\phi_f^a(x)^{\bullet}\,\partial_\nu\phi_f^b(x)^{\bullet}]^*=\langle 0;f|T^*\frac{\partial}{\partial x_\mu}\phi_f^a(x)\frac{\partial}{\partial y_\nu}\phi_f^b(y)|0;f\rangle$$
$$=-i\frac{\partial}{\partial x_\mu}\frac{\partial}{\partial y_\nu}\Delta_F(x-y)\delta_{ab} \tag{2.2b}$$

Here the notation is such that a state $|\ldots;f\rangle$ means either an "in" ($f=\text{in}$) or an "out" ($f=\text{out}$) state. We see that (2.2a) and (2.2b) differ by the noncovariant (normal-dependent) term $n_\mu n_\nu \delta_4(x-y)$. Equations (2.2) suggest the definition of pure noncovariant contractions for ϕ_f^a fields as

$$
\begin{aligned}
[\phi_f^a(x)^\bullet \phi_f^b(y)^\bullet]^n &= 0 \\
[\partial_\mu \phi_f^a(x)^\bullet \phi_f^b(y)^\bullet]^n &= [\phi_f^a(x)^\bullet\, \partial_\nu \phi_f^b(y)^\bullet]^n = 0 \\
[\partial_\mu \phi_f^a(x)^\bullet\, \partial_\nu \phi_f^b(y)^\bullet]^n &= -i n_\mu n_\nu \delta_{ab} \delta_4(x-y)
\end{aligned}
\tag{2.3}
$$

with other pure noncovariant contractions given in Appendix A.

Relations (2.2) and (2.3) suggest that we write the T product as (Šoln, 1973)

$$T = T^* T_n \tag{2.4}$$

where T_n, a symbolic notation for the noncovariant T product, means: When T_n acts on a product involving ϕ_f operators, one sums up all possible (including the zeroth) pure noncovariant contractions according to (where $1_f, 2_f, \ldots,$ is a short-hand notation for free ϕ_f operators)

$$
\begin{aligned}
T_n(1_f 2_f \cdots r_f) &= 1_f 2_f \cdots r_f + [1_f^\bullet 2_f^\bullet]^n 3_f \cdots r_f \\
&\quad + [1_f^\bullet 3_f^\bullet]^n 2_f \cdots r_f + \cdots + [1_f^\bullet 2_f^\bullet]^n [3_f^\bullet 4_f^\bullet]^n 5_f \cdots r_f + \cdots
\end{aligned}
\tag{2.5}
$$

[In Šoln (1973) the symbol T_n is equivalent to T^*T_n here. This is why T_n products from that work and here differ by T^* products.] The action of the inverse of T_n, T_n^{-1}, clearly is

$$
\begin{aligned}
T_n^{-1}(1_f 2_f \cdots r_f) &= 1_f 2_f \cdots r_f - [1_f^\bullet 2_f^\bullet]^n 3_f \cdots r_f \\
&\quad - [1_f^\bullet 3_f^\bullet]^n 2_f \cdots r_f - \cdots \\
&\quad + (-)^2 [1_f^\bullet 2_f^\bullet]^n [3_f^\bullet 4_f^\bullet]^n 5_f \cdots r_f + \cdots
\end{aligned}
\tag{2.6}
$$

One easily verifies that $T_n T_n^{-1}(1_f 2_f \cdots) = T_n^{-1} T_n(1_f 2_f \cdots) = 1_f 2_f \cdots$, which on the formal level can be stated as

$$T_n^s T_n^{-s} = 1, \qquad s = \pm 1 \tag{2.7}$$

where, for carrying out actual pure noncovariant contractions, we have introduced the notation

$$T_n^s = \exp[s\hat{N}_f], \qquad s = \pm 1 \tag{2.8a}$$

Here, the action of $\hat{N}_f$ means summation over only bilinear pure noncovariant contractions ($s = \pm 1$):

$$
\begin{aligned}
s\hat{N}_f(1_f 2_f \cdots r_f) &= s[1_f^\bullet 2_f^\bullet]^n 3_f \cdots r_f + \cdots + s[(r-1)_f^\bullet r_f^\bullet]^n 1_f \cdots (r-2)_f \\
\frac{s^2}{2!} \hat{N}_f(\hat{N}_f 1_f 2_f \cdots r_f) &= [1_f^\bullet 2_f^\bullet]^n [3_f^\bullet 4_f^\bullet]^n 5_f \cdots r_f + \cdots, \quad \text{etc.}
\end{aligned}
\tag{2.8b}
$$

One easily verifies the equivalence of relations (2.8) with relations (2.5) and (2.6). Now, with the help of (2.8a), from (2.4) we obtain

$$T^* = TT_n^{-1} \tag{2.9}$$

We see that while relation (2.4) yields a T^* product from a T product after carrying out T_n contractions, relation (2.9) yields a T product from a T^* product after carrying out T_n^{-1} contractions.

A note of caution: While the factors within T and T^* products can be rearranged at will (taking into account the statistics of field operators), within a T_n^s product this cannot be done if a T_n^s product stands alone. However, we can also rearrange the factors within a T_n^s product, if T_n^s multiplies T or T^* from the right [compare with (2.4) and (2.9)].

Next, since the T^* product is evaluated by means of pure covariant T^* contractions, we have to demand

$$T_n^s T^*(1_f 2_f \cdots r_f) = T^*(1_f 2_f \cdots r_f), \qquad s = \pm 1 \tag{2.10}$$

or symbolically

$$T_n^s T^* = T^*, \qquad s = \pm 1 \tag{2.11}$$

But, applying T_n^s to T^* from (2.9), we see that we must also have

$$T_n^s T = T, \qquad s = \pm 1 \tag{2.12}$$

Applying T to relation (2.9) and taking into account that formally $TT = T$, we obtain

$$TT^* = T^* \tag{2.13}$$

Taking the square of relation (2.9), with (2.12) and (2.13) taken into account, we also deduce that

$$T^*T^* = T^* \tag{2.14}$$

Combining (2.14) with (2.4), we finally have

$$T^*T = T \tag{2.15}$$

Relations (2.11)–(2.15) are simply invariance statements for T and T^* products under the actions of T_n^s, T, and T^*, respectively; the only way the T and T^* products can be transformed into each other is through relations (2.4) and (2.9). However, this is not the end of the story. Namely, we know that T and T^* products are expressible in terms of normal-ordered products through T and T^* contractions, respectively. Denoting symbolically a normal-ordered product by ::, from relations (2.11)–(2.15) we conclude that :: formally has the following properties:

$$T::=:: \tag{2.16}$$

$$T^*::=:: \tag{2.17}$$

$$T_n^s::=::, \qquad s=\pm 1 \tag{2.18}$$

These relations are also simply invariance statements for the normal-ordered product under the actions of T, T^*, and T_n^s.

Next we write down some rules for evaluating T_n^s $(s=\pm 1)$ products. Let us introduce a common symbol, $T^{(\alpha)}$ $(\alpha=0, *, s=\pm 1, 2, 3)$, for all products:

$$\begin{aligned} &T^{(0)}=T, \qquad T^{(*)}=T^*, \qquad T^{(s)}=T_n^s \\ &T^{(2)}=::, \qquad T^{(3)}=1 \end{aligned} \tag{2.19}$$

where, for example, $T^{(2)}1_f2_f=:1_f2_f:$, $T^{(3)}1_f2_f=1_f2_f$, etc. Let $A_f=1_f2_f\cdots$, $B_f=1'_f2'_f\cdots$, etc.; then specifically

$$T_n^sA_fB_f=T_n^s(A_f;B_f)(T_n^sA_f)(T_n^sB_f), \qquad s=\pm 1 \tag{2.20}$$

Here $T_n^s(A_f;B_f)$ generates T_n^s contractions (starting with the zeroth contraction) only between A_f and B_f, and, consistent with relations (2.8), it can be written as

$$T_n^s(A_f;B_f)=\exp[s\hat{N}(A_f;B_f)], \qquad s=\pm 1 \tag{2.21a}$$

where, in general,

$$\begin{aligned} &s\hat{N}(A_f;B_f)(T^{(\alpha)}A_f)(T^{(\beta)}B_f) \\ &\quad =s[1_f^{\cdot}1_f^{\cdot\cdot}]^n(T^{(\alpha)}2_f3_f\cdots)(T^{(\beta)}2'_f3'_f\cdots)+\cdots \\ &\frac{s^2}{2!}\hat{N}^2(A_f;B_f)(T^{(\alpha)}A_f)(T^{(\beta)}B_f) \\ &\quad =[1_f^{\cdot}1_f^{\cdot\cdot}]^n[2_f^{\cdot}2_f^{\cdot\cdot}]^n(T^{(\alpha)}3_f4_f\cdots)(T^{(\beta)}3'_f4'_f\cdots)+\cdots, \quad \text{etc.} \\ &s=\pm 1 \end{aligned} \tag{2.21b}$$

We extend relation (2.20) by adding another factor C_f:

$$T_n^sA_fB_fC_f=T_n^s(A_fB_f;C_f)(T_n^sA_fB_f)(T_n^sC_f) \tag{2.22a}$$

$$=T_n^s(A_f;B_fC_f)(T_n^sA_f)(T_n^sB_fC_f) \tag{2.22b}$$

$$\begin{aligned} &\equiv T_n^s(A_f;B_f)T_n^s(A_f;C_f)T_n^s(B_f;C_f) \\ &\quad \times T_n^s(A_f)(T_n^sB_f)(T_n^sC_f), \qquad s=\pm 1, \end{aligned} \tag{2.22c}$$

giving

$$T_n^s(A_f; B_fC_f) = T_n^s(B_fC_f; A_f)$$
$$= T_n^s(A_f; B_f)T_n^s(A_f; C_f), \qquad s = \pm 1 \tag{2.23a}$$
$$T_n^s(A_f; B_f) = T_n^s(B_f; A_f), \qquad s = \pm 1 \tag{2.23b}$$

Relations (2.23) give a recipe for writing down the T_n^s product with an arbitrary number of factors. As a consequence of relations (2.11), (2.12), and (2.18), we have the following important relations ($s = \pm 1$)

$$T_n^s(TA_f)(TB_f)C_f = T_n^s(A_f; B_fC_f)T_n^s(B_f; C_f)$$
$$\times (TA_f)(TB_f)(TC_f) \tag{2.24a}$$
$$T_n^s(T^*A_f)(T^*B_f)C_f = T_n^s(A_f; B_fC_f)T_n^s(B_f; C_f)$$
$$\times (T^*A_f)(T^*B_f)(T_n^sC_f) \tag{2.24b}$$
$$T_n^s(TA_f)(T^*B_f)C_f = T_n^s(A_f; B_fC_f)T_n^s(B_f; C_f)$$
$$\times (TA_f)(T^*B_f)(T_n^sC_f) \tag{2.24c}$$
$$T_n^s{:}A_f{::}B_f{:}C_f = T_n^s(A_f; B_fC_f)$$
$$\times T_n^s(B_f; C_f){:}A_f{::}B_f{:}(T_n^sC_f) \tag{2.24d}$$
$$T_n^s{:}A_f{:}(TB_f)C_f = T_n^s(A_f; B_fC_f)$$
$$\times T_n^s(B_f; C_f){:}A_f{:}(TB_f)(T_n^sC_f), \quad \text{etc.} \tag{2.24e}$$

Relations (2.24) explicitly exhibit the fact that T_n^s does not generate pure noncovariant contractions within T_n^s, T, T^*, or :: products.

We wish to argue that because the symbol T can be inserted at arbitrary places within the T product, the symbol T^* can also be inserted at arbitrary places within the T^* product. To show this, we look first at

$$T^*A_fB_f = TT_n^{-1}A_fB_f = TT_n^{-1}(A_f; B_f)(T_n^{-1}A_f)(T_n^{-1}B_f) \tag{2.25a}$$
$$= TT_n^{-1}(A_f; B_f)(T_n^{-1}A_f)T(T_n^{-1}B_f) \tag{2.25b}$$

where in (2.25a) we took into account (2.20), while in (2.25b) we inserted the symbol T into all terms with pure noncovariant (including the zeroth) contractions generated by $T_n^{-1}(A_f; B_f)$. On the other hand, we can look at

$$T^*A_fT^*B_f = TT_n^{-1}A_f(TT_n^{-1}B_f) \tag{2.26a}$$
$$= TT_n^{-1}(A_f; B_f)(T_n^{-1}A_f)T(T_n^{-1}B_f) \tag{2.26b}$$

where in (2.26b) we took into account that $T_n^{-1}T = T$. Comparison of (2.25b) with (2.26b) yields

$$T^*A_fB_f = T^*A_fT^*B_f \tag{2.27}$$

By changing, for example, $B_f \to B_f C_f$ in (2.27), one easily argues that indeed the T^* symbol can be inserted at arbitrary places within the T^* product itself. Actually, this should not be surprising, since the T^* symbol can be viewed as a covariant time-ordered operator, as opposed to T, which is the usual (noncovariant) time-ordering operator.

It should not be surprising that for the T product we can write down relations similar to relations (2.20), (2.22), and (2.23). The simplest one is

$$TA_fB_f = T(A_f; B_f)(TA_f)(TB_f) \tag{2.28}$$

One interpretation for $T(A_f; B_f)$ in (2.28) is that, after having done T contractions (including the zeroth ones) within A_f and B_f, one still has to do all possible (including the zeroth) contractions between A_f and B_f when reducing TA_fB_f into a sum of normal products. Another interpretation of $T(A_f; B_f)$ is that, after the factors within A_f and B_f have been time-ordered, $T(A_f; B_f)$ now time-orders $(TA_f)(TB_f)$, in such a way, however, that the relative orders of factors from TA_f and the relative order of factors from TB_f do not change. This interpretation will be very useful later on. Clearly we can generalize (2.28) by adding another factor,

$$TA_fB_fC_f = T(A_fB_f; C_f)(TA_fB_f)(TC_f) \tag{2.29a}$$

$$= T(A_f; B_fC_f)(TA_f)(TB_fC_f) \tag{2.29b}$$

$$T(A_f; B_fC_f) = T(B_fC_f; A_f) = T(A_f; B_f)T(A_f; C_f) \tag{2.29c}$$

$$T(A_f; B_f) = T(B_f; A_f) \tag{2.29d}$$

and so on.

Combining the relation (2.20) with (2.28) and relations (2.22) and (2.23) with (2.29), we immediately have

$$T^*A_fB_f = T^*(A_f; B_f)(T^*A_f)(T^*B_f) \tag{2.30}$$

$$T^*A_fB_fC_f = T^*(A_fB_f; C_f)(T^*A_fB_f)(T^*C_f) \tag{2.31a}$$

$$= T^*(A_f; B_fC_f)(T^*A_f)(T^*B_fC_f), \quad \text{etc.} \tag{2.31b}$$

where

$$T^*(A_f; B_f) = T(A_f; B_f)T_n^{-1}(A_f; B_f) \tag{2.32a}$$

$$T^*(A_f; B_fC_f) = T^*(B_fC_f; A_f) = T^*(A_f; B_f)T^*(A_f; C_f) \tag{2.32b}$$

$$T^*(A_f; B_f) = T^*(B_f; A_f) \tag{2.32c}$$

Here we can make similar interpretations for T^* products. Clearly, relation (2.30) means that after having done T^* contractions (including the zeroth ones) within A_f and B_f, $T^*(A_f; B_f)$ generates all possible contractions (including the zeroth one) between A_f and B_f when reducing $T^*A_fB_f$ into a sum of normal products. On the other hand, we can call $T^* = TT_n^{-1}$ a covariant time-ordering operator. Then the interpretation of $T^*(A_f; B_f)$ is that, after having covariantly time-ordered factors within A_f and B_f, $T^*(A_f; B_f)$, now covariantly time-order $(T^*A_f)(T^*B_f)$ with the stipulation that the relative orders of factors from T^*A_f, as well as from T^*B_f, do not change.

Finally, let us discuss T and T^* products at single space-time point x. Suppose we have $Q_f(x, y) = (\partial_\mu \phi_f(x))(\partial_\nu \phi_f(y))$. Then, because a T product is undefined at equal times, it is ambiguous as to what $TQ_f(x)$ is, where $Q_f(x) \equiv Q_f(x, x)$. However, we can look for some physical principles to define in general $TA_f(x)$, where $A_f(x)$ depends on $\phi_f(x)$, $\partial_\mu \phi_f(x)$, $\partial_\mu \partial_\nu \phi_f(x), \ldots$. In the next section we shall see that the definition

$$TA_f(x) := A_f(x) \tag{2.33}$$

is consistent with the requirement that the S matrix derived from the canonical formalism be consistent with our explicitly derived covariant S matrix. Next, applying either T^* or T_n^s ($s = \pm 1$) to (2.33), with the help of (2.15) and (2.12), we also obtain

$$T^*A_f(x) = A_f(x) \tag{2.34}$$

$$T_n^s A_f(x) = A_f(x), \qquad s = \pm 1 \tag{2.35}$$

The immediate consequence of relation (2.35) is that, with noncovariant contractions as outlined in (2.2a) and in Appendix A, it generally requires

$$\delta_4(0) = 0 \tag{2.36a}$$

$$(\partial_\mu \delta_4)(0) = 0, \qquad (\partial_\mu \partial_\nu \delta_4)(0) = 0, \ldots \tag{2.36b}$$

For example, (2.36a) immediately follows if one chooses for $A_f(x)$, $(\partial_\mu \phi_a(x))(\partial_\nu \phi_b(x))$ [compare with (2.2a)] or $\phi_a(x)\, \partial_\mu \partial_\nu \phi_b(x)$ (compare with Appendix A). Relation (2.36a) has been proven by means of dimensional regularization (Simon, 1990; Barua and Gupta, 1977; Capper and Liebbrandt, 1973, 1974; Tataru, 1975; 't Hooft and Veltman, 1972). However, as shown in Appendix B, the dimensional regularization also gives relations (2.36b). We shall assume the validity of relations (2.33)–(2.36) throughout this article.

3. COVARIANT PDECC FORMALISM INVOLVING LAGRANGIAN DENSITY ONLY

Our aim is to develop the covariant PDECC formalism for the S matrix and the interpolating (Heisenberg) fields when the Lagrangian density is of the form

$$\mathscr{L}(x) = \mathscr{L}_0(x) + \mathscr{L}_{\text{int}}(x) \equiv \tilde{\mathscr{L}}(x; \mathbf{g}) \tag{3.1a}$$

$$\mathscr{L}_0(x) \equiv \mathscr{L}_0(\phi(x), \partial_\mu \phi(x)) \tag{3.1b}$$

$$\begin{aligned} \mathscr{L}_{\text{int}}(x) &\equiv \mathscr{L}_{\text{int}}(\phi(x), \partial_\mu \phi(x), \partial_\mu \partial_\nu \phi(x), \ldots ; \mathbf{g}) \\ &\equiv \tilde{\mathscr{L}}_{\text{int}}(x; \mathbf{g}), \qquad \tilde{\mathscr{L}}_{\text{int}}(x, \mathbf{0}) = 0 \end{aligned} \tag{3.1c}$$

Here $\phi(x)$ denotes a set of independent interpolating fields corresponding to the system of interacting particles; they are assumed to interact through many independent interactions which are characterized by coupling constants $g_1, g_2, \ldots$, denoted here as vector $\mathbf{g} = (g_1, g_2, \ldots)$. The asymptotic free field operators $\phi_f(x)$ $[=\phi_{\text{in}}(x), \phi_{\text{out}}(x)]$ correspond also to a similar set of independent asymptotic free field operators. Now, consistent with the adiabatic switching on and off of the interaction, to each $\mathscr{L}_{\text{int}}(x_i)$, or equivalently, to each coupling constant vector $\mathbf{g}$, we attach an adiabatic factor

$$e(x_i) = \exp[-\varepsilon |n \cdot x_i|], \qquad \varepsilon \to +0 \tag{3.1d}$$

where the $\varepsilon \to +0$ limit is taken after pertinent manipulations have been carried out. When necessary, the adiabatic factor will be written out explicitly. Before we undertake the covariant PDECC formulation for interpolating fields and the S matrix, let us specify what one expects from the S matrix. If we have (f = in, out)

$$\begin{aligned} F_f(x, y, \ldots) = F(&\phi_f(x), \partial_{\mu_1}\phi_f(x), \partial_{\mu_1}\partial_{\mu_2}\phi_f(x), \ldots, \\ &\phi_f(y), \partial_{\nu_1}\phi_f(y), \partial_{\nu_1}\partial_{\nu_2}\phi_f(y), \ldots ; \mathbf{g}) \end{aligned} \tag{3.2}$$

expressible as a Taylor series in its arguments and in which the differences between any of the times $t_x, t_y, \ldots$, are finite, then we should have

$$F_{\text{out}}(x, y, \ldots) = S^\dagger(\mathbf{g}) F_{\text{in}}(x, y, \ldots) S(\mathbf{g}) \tag{3.3}$$

On the other hand, if we time-order $F_{\text{in}}(x, y, \ldots)$ in (3.3), then the left side of (3.3) becomes time-ordered also, which means

$$TF_{\text{out}}(x, y, \ldots) = S^\dagger(\mathbf{g})(TF_{\text{in}}(x, y, \ldots)) S(\mathbf{g}) \tag{3.4}$$

Suppose now that at the beginning $F_f(x, y, \ldots)$ did not depend on the derivatives of ϕ_f's. Then with appropriate differentiations of (3.4) [compare with (2.1)], we can achieve

$$T^* F_{\text{out}}(x, y, \ldots) = S^\dagger(\mathbf{g})(T^* F_{\text{in}}(x, y, \ldots)) S(\mathbf{g}) \tag{3.5}$$

Clearly, relations (3.4) and (3.5) imply each other; this can be seen by, say, using (2.9) in (3.5) and carrying out pure noncovariant contractions that are generated by T_n^{-1}. Because of the unitarity of the S matrix, these pure noncovariant contractions are the same on both sides of relation (3.5) [compare with (2.21)]. Finally, using covariant T^* contractions, which, because of the unitarity of the S matrix, are the same on both sides of (3.5), we expand both sides of (3.5) in terms of normal-ordered products; this clearly implies relation (3.3).

After these preliminaries, we now turn to the formulation of the covariant PDECC for the S matrix and the interpolating fields. The way one usually defines the interpolating field $\phi(x)$ is through the relation

$$\phi(x)=S^\dagger(\mathbf{g})T(S(\mathbf{g})\phi_{\text{in}}(x))$$

where clearly $\phi\to\phi_{\text{in,out}}$ as $t_x\to-\infty, +\infty$. However, since the T product is not generally a covariant quantity, this definition may yield a noncovariant $\phi(x)$. Therefore, we start with an explicitly covariant definition for the interpolating field as

$$\phi(x)=S^\dagger(\mathbf{g})T^*(S(\mathbf{g})\phi_{\text{in}}(x)) \tag{3.6a}$$

$$=S^\dagger(\mathbf{g})T^*(S(\mathbf{g});\,\phi_{\text{in}}(x))S(\mathbf{g})\phi_{\text{in}}(x) \tag{3.6b}$$

where in (3.6b) we took into account relation (2.30) and we assumed that the S matrix is expressible either as a T or T^* product, so that $T^*S=S$ [compare with relations (2.14) and (2.15)]. Since $S(\mathbf{0})=1$, we have the important relations

$$\phi_{\text{in}}(x)=\phi(x)|_{\mathbf{g}=0} \tag{3.7a}$$

$$\frac{\partial\phi_{\text{in}}(x)}{\partial g_i}=0 \tag{3.7b}$$

Next, with $\phi_1^{\text{in}}(x)$, $\phi_2^{\text{in}}(x), \ldots$, denoting specific fields, we look at

$$\begin{aligned} &S^\dagger(\mathbf{g})T^*S(\mathbf{g})\phi_1^{\text{in}}(x)\phi_2^{\text{in}}(y)\\ &\quad=T^*(\phi_1^{\text{in}}(x);\,\phi_2^{\text{in}}(y))S^\dagger(\mathbf{g})T^*(S(\mathbf{g});\,\phi_1^{\text{in}}(x))S(\mathbf{g})\phi_1^{\text{in}}(x)\\ &\qquad\times S^\dagger(\mathbf{g})T^*(S(\mathbf{g});\,\phi_2^{\text{in}}(y))S(\mathbf{g})\phi_2^{\text{in}}(y) \end{aligned} \tag{3.8a}$$

$$=T^*(\phi_1^{\text{in}}(x);\,\phi_2^{\text{in}}(y))\phi_1(x)\phi_2(y)\equiv T^*\phi_1(x)\phi_2(y) \tag{3.8b}$$

where in (3.8a) we took into account relations (2.30)–(2.32), the fact that $T^*S=S$, inserted $SS^\dagger=1$ between $T^*(S;\,\phi_1^{\text{in}})$ and $T^*(S;\,\phi_2^{\text{in}})$, and acknowledged relations (3.6). Here we interpret all T^*'s as covariant time-ordering operators [see the discussion after relations (2.30)–(2.32)]; this allows us to pull $T^*(\phi_1^{\text{in}};\,\phi_2^{\text{in}})$ all the way to the left. It is obvious that if in (3.8a) we had started with $T^*\phi_1^{\text{in}}(x)\phi_2^{\text{in}}(y)$, the result would have been the same.

If in relations (3.8) we start with $\partial_{\mu_1} \cdots \partial_{\mu_n} \phi_1^{\text{in}}(x)$ and $\partial_{\nu_1} \cdots \partial_{\nu_m} \phi_2^{\text{in}}(y)$ instead of with $\phi_1^{\text{in}}(x)$ and $\phi_2^{\text{in}}(y)$, we obtain

$$
\begin{aligned}
&T^* \partial_{\mu_1}(x) \cdots \partial_{\mu_n}(x) \phi_1(x)\, \partial_{\nu_1}(y) \cdots \partial_{\nu_m}(y) \phi_2(y) \\
&\quad = S^\dagger(\mathbf{g}) T^* S(\mathbf{g})\, \partial_{\mu_1}(x) \cdots \partial_{\mu_n}(x) \phi_1^{\text{in}}(x)\, \partial_{\nu_1}(y) \cdots \partial_{\nu_m}(y) \phi_2^{\text{in}}(y) \qquad (3.9)
\end{aligned}
$$

On the other hand, we can differentiate directly relations (3.8). Then consistent with (2.1), we see that in general the definition of the T^* product for interpolating fields is

$$
\begin{aligned}
&T^*(\partial_{\mu_1}(x) \cdots \partial_{\mu_n}(x) \phi_1(x)\, \partial_{\nu_1}(y) \cdots \partial_{\nu_m}(y) \phi_2(y) \cdots) \\
&\quad = (\partial_{\mu_1}(x) \cdots \partial_{\mu_n}(x))_1 (\partial_{\nu_1}(y) \cdots \partial_{\nu_m}(y))_2 \cdots T^*(\phi_1(x) \phi_2(y) \cdots) \quad (3.10)
\end{aligned}
$$

where the indices attached to the derivatives indicate the fields on which these derivatives act. With [compare with (3.21)]

$$
\begin{aligned}
&F(x, y, \ldots) \\
&\quad = F(\phi(x), \partial_{\mu_1}\phi(x), \partial_{\mu_1}\partial_{\mu_2}\phi(x), \ldots, \phi(y), \partial_{\nu_1}\phi(y), \partial_{\nu_1}\partial_{\nu_2}\phi(y), \ldots; \mathbf{g}) \qquad (3.11)
\end{aligned}
$$

the generalization of relations (3.8) and (3.9) is

$$
T^* F(x, y, \ldots) = S^\dagger(\mathbf{g}) T^*(S(\mathbf{g}) F_{\text{in}}(x, y, \ldots)) \qquad (3.12)
$$

Now let us look at

$$
\begin{aligned}
&S^\dagger(\mathbf{g}) T^* S(\mathbf{g}) \phi_{\text{in}}(x)\, \partial_\mu \partial_\nu \phi_{\text{in}}(x) \\
&\quad = S^\dagger(\mathbf{g}) T^*(\phi_{\text{in}}(x); \partial_\mu \partial_\nu \phi_{\text{in}}(x)) T^*(S(\mathbf{g}); \phi_{\text{in}}(x)) S(\mathbf{g}) \phi_{\text{in}}(x) \\
&\qquad \times S^\dagger(\mathbf{g}) T^*(S(\mathbf{g}); \partial_\mu \partial_\nu \phi_{\text{in}}(x)) S(\mathbf{g})\, \partial_\mu \partial_\nu \phi_{\text{in}}(x) \qquad (3.13a) \\
&\quad = \phi(x)\, \partial_\mu \partial_\nu \phi(x) \qquad (3.13b)
\end{aligned}
$$

Here we took into account that $T^*(\phi_{\text{in}}(x); \partial_\mu \partial_\nu \phi(x)) = 1$ [compare with (2.34)] and that $T^* S = S$, have inserted $SS^\dagger = 1$ between $T^*(S; \phi_{\text{in}})$ and $T^*(S; \partial_\mu \partial_\nu \phi_{\text{in}})$, and, of course, took into account relations (3.6). Clearly, from (3.13) we obtain easily by induction

$$
A(x) = S^\dagger(\mathbf{g}) T^* S(\mathbf{g}) A_{\text{in}}(x) \qquad (3.14a)
$$

$$
A_f(x) = A(\phi_f(x), \partial_\mu \phi_f(x), \partial_\mu \partial_\nu \phi_f(x), \ldots; \mathbf{g}) \qquad (3.14b)
$$

$$
A(x) = A(\phi(x), \partial_\mu \phi(x), \partial_\mu \partial_\nu \phi(x), \ldots; \mathbf{g}) \qquad (3.14c)
$$

$$
T^* A(x) = A(x) \qquad (3.14d)
$$

where relation (3.14d) follows by comparing (3.14a) with (3.12). It is obvious that relation (3.14d) is a direct consequence of $T^*A_f(x)=A_f(x)$ [relation (2.34)].

Finally from relation (3.12), we have the following important special case:

$$S(\mathbf{g})T^*(F(x)G(y)\cdots)=T^*(S(\mathbf{g})F_{\text{in}}(x)G_{\text{in}}(y)\cdots) \tag{3.15}$$

where $F, G, \ldots$ $(F_{\text{in}}, G_{\text{in}}, \ldots)$ depend locally on ϕ's (ϕ_{in}'s) and their derivatives.

In order to ensure the unitarity of the S matrix, the PDECC for it is taken to be (Šoln, 1973)

$$\frac{1}{i}\frac{\partial}{\partial g_i}S(\mathbf{g})=S(\mathbf{g})\int d^4x\,\mathscr{L}_i^*(x) \tag{3.16a}$$

$$=\int d^4x\,T^*S(\mathbf{g})\mathscr{L}_{i,\text{in}}^*(x) \tag{3.16b}$$

where (3.16b) is the consequence of applying (3.14a) to (3.16a), and where [compare with (3.1)]

$$\begin{aligned}\mathscr{L}_i^*(x)&\equiv\frac{\partial^*}{\partial g_i}\mathscr{L}(x)=\frac{\partial^*}{\partial g_i}\mathscr{L}_{\text{int}}(x)\\ \mathscr{L}_{i,\text{in}}^*(x)&=\mathscr{L}_i^*(x)|_{\phi=\phi_{\text{in}}}\end{aligned} \tag{3.17}$$

The "star" partial derivative acts as an ordinary derivative on the coupling constant-dependent coefficients that multiply $\phi(x)$, $\partial_\mu\phi(x), \ldots,$ or $\phi_f(x)$, $\partial_\mu\phi_f(x), \ldots,$ in such a way, however, that

$$\frac{\partial^*\phi(x)}{\partial g_i}=0, \qquad \frac{\partial^*\phi_f(x)}{\partial g_i}=0 \tag{3.18a}$$

$$\left[\frac{\partial^*}{\partial g_i}, \frac{\partial}{\partial x_\mu}\right]=0 \tag{3.18b}$$

A simpler version of the PDECC for the S matrix is obtained if we "freeze" the physical coupling constants in $\mathscr{L}_{\text{int}}$. Their place is now taken by a single "mathematical" dimensionless coupling constant λ, which is varied between 0 and 1. Relations (3.1a) and (3.16) are now written, respectively, as

$$\mathscr{L}(x)=\mathscr{L}_0(x)+\lambda\mathscr{L}_{\rm int}(x) \tag{3.19}$$

$$\frac{1}{i}\frac{d}{d\lambda}S(\lambda)=\int d^4x\, S(\lambda)\mathscr{L}_{\rm int}(x) \tag{3.20a}$$

$$=\int d^4x\, T^*S(\lambda)\mathscr{L}_{\rm int}^{\rm in}(x) \tag{3.20b}$$

Of course, the solutions of (3.16) and (3.20), as we shall see later, are fully compatible with each other.

Next, in order to show the correctness of $T_n{}^sA_f(x)=A_f(x)$, $s=\pm 1$, we now specialize $\mathscr{L}_{\rm int}$ as

$$\mathscr{L}_{\rm int}(x)=\mathscr{L}_{\rm int}(\phi(x),\, \partial_\mu\phi(x);\, \mathbf{g}) \tag{3.21a}$$

However, from the PDECC in conjunction with the canonical formalism (Sŏln, 1973) (see also Appendix C), we have

$$\frac{1}{i}\frac{d}{d\lambda}S(\lambda)=\int d^4x\, TS(\lambda)\mathscr{L}_{\rm int}(\phi_{\rm in}(x),\, (\partial_\mu\phi(x))_{\rm in};\, \mathbf{g}) \tag{3.21b}$$

In Appendix C we show that

$$\mathscr{L}_{\rm int}(\phi_{\rm in}(x),\, (\partial_\mu\phi(x))_{\rm in};\, \mathbf{g})=\mathscr{L}_{\rm int}(\phi_{\rm in}(x),\, \partial_\mu\phi_{\rm in}(x);\, \mathbf{g})+O(\lambda) \tag{3.22}$$

which, after equating (3.21b) with (3.20b) at $\lambda=0$, yields

$$\mathscr{L}_{\rm int}(\phi_{\rm in}(x),\, \partial_\mu\phi_{\rm in}(x);\, \mathbf{g})=T_n\mathscr{L}_{\rm int}(\phi_{\rm in}(x),\, \partial_\mu\phi_{\rm in}(x);\, \mathbf{g}) \tag{3.23}$$

where $T=T^*T_n$ was taken into account. Relation (3.23) is also obtainable from more general relations (at $\lambda=0$) that connect interaction Hamiltonian and Lagrangian densities and which will be discussed in Section 4. Relation (3.23) is automatically satisfied if $\mathscr{L}_{\rm int}$ is in a normal form [compare with (2.18)]. However, $\mathscr{L}_{\rm int}$ is not always written in a normal form; an example is a chiral invariant Lagrangian (Sŏln, 1973; Gerstein *et al.*, 1971; Weinberg, 1968). Thus, in general, (3.23) implies relation (2.36a), which is consistent with the dimensional regularization. In turn, dimensional regularization implies also relation (2.36b), which, together with (2.36a), implies $T_n{}^sA_f(x)=A_f(x)$, $s=\pm 1$, for any local free field function $A_f(x)$ [relation (2.35)]. As we see, the requirement that the S matrix be simultaneously described with the explicitly covariant PDECC and the PDECC that incorporates the canonical formalism essentially requires relations (2.33)–(2.36).

Next we write down the covariant PDECC involving the S matrix and arbitrary field quantities. Utilizing relations (3.14)–(3.16), we have

$$\frac{1}{i}\frac{\partial}{\partial g_i} S(\mathbf{g})T^*F(x)G(y)\cdots Q(z)$$

$$=\frac{1}{i}\frac{\partial}{\partial g_i} T^*S(\mathbf{g})F_{\text{in}}(x)G_{\text{in}}(y)\cdots Q_{\text{in}}(z) \tag{3.24a}$$

$$=\int d^4w\, S(\mathbf{g})T^*F(x)G(y)\cdots Q(z)\mathscr{L}_i^*(w)$$

$$+S(\mathbf{g})T^*\frac{1}{i}\frac{\partial^*}{\partial g_i}F(x)G(y)\cdots Q(z) \tag{3.24b}$$

where relations (3.15)–(3.18) and (2.27) were taken into account. Relations (3.24) immediately yield the covariant PDECC for field quantities

$$\frac{1}{i}\frac{\partial}{\partial g_i}T^*F(x)G(y)\cdots Q(z)=\int d^4w\,\{T^*(F(x)G(y)\cdots Q(z)\mathscr{L}_i^*(w))$$

$$-\mathscr{L}_i^*(w)T^*(F(x)G(y)\cdots Q(z))$$

$$+T^*\frac{1}{i}\frac{\partial^*}{\partial g_i}(F(x)G(y)\cdots Q(z)) \tag{3.25}$$

In order to write down the explicitly covariant expression for the S matrix, let us look at

$$\left(\frac{1}{i}\right)^2\frac{\partial^2}{\partial g_i\partial g_j}S(\mathbf{g})=\int d^4x\,d^4y\,S(\mathbf{g})T^*\mathscr{L}_i^*(x)\mathscr{L}_j^*(y)$$

$$+\frac{1}{i}\int d^4x\,S(\mathbf{g})\mathscr{L}_{i,j}^{2*}(x) \tag{3.26}$$

where we used (3.24a) and introduced the notation

$$\mathscr{L}_{i_1,i_2,\ldots,i_n}^{n*}(x)=\frac{\partial^{*n}\mathscr{L}(x)}{\partial g_{i_1}\partial g_{i_2}\cdots\partial g_{i_n}} \tag{3.27}$$

Relations (3.16) and (3.25) evaluated at $\mathbf{g}=0$ yield the S matrix as a power series up to second order in coupling constants. Continuing to higher orders yields the result

$$S(\mathbf{g}) = T^* \exp\left\{ i \int d^4x \sum_{n=1}^{\infty} \frac{1}{n!} \left[\left(\sum_i g_i \frac{\partial^*}{\partial g_i'} \right)^n \tilde{\mathscr{L}}(x; \mathbf{g}') \right]_{\mathbf{g}'=0} \right\} \tag{3.28}$$

Consistent with (3.7) and (3.1), we can write in (3.28)

$$\mathscr{L}^{n*}_{i_1, i_2, \ldots, i_n}(x)|_{\mathbf{g}=0} = \mathscr{L}^{n*}_{i_1, i_2, \ldots, i_n; \mathrm{in}}(x)|_{\mathbf{g}=0} \tag{3.29}$$

so that the exponent of (3.28) becomes

$$\sum_{n=1}^{\infty} \frac{1}{n!} \left[\left(\sum_i g_i \frac{\partial^*}{\partial g_i} \right)^n \tilde{\mathscr{L}}(x; \mathbf{g}') \right]_{\mathbf{g}'=0}$$

$$= \sum_{n=1}^{\infty} \frac{1}{n!} \left[\left(\sum_i g_i \frac{\partial^*}{\partial g_i} \right)^n \tilde{\mathscr{L}}(x; \mathbf{g}')_{\mathrm{in}} \right]_{\mathbf{g}'=0} = \tilde{\mathscr{L}}_{\mathrm{int;in}}(x; \mathbf{g}) \tag{3.30}$$

Therefore, the solution of (3.20),

$$S(\lambda) = T^* \exp\left[i\lambda \int d^4x \, \mathscr{L}_{\mathrm{int;in}}(x) \right] \tag{3.31a}$$

$$= T\left\{ T_n^{-1} \exp\left[i\lambda \int d^4x \, \mathscr{L}_{\mathrm{int;in}}(x) \right] \right\} \tag{3.31b}$$

is also the solution of (3.28) at $\lambda = 1$.

It is clear that in both manifestly covariant expressions (3.28) and (3.31a) the S matrix obeys the original Matthews theorem (Simon, 1990; Matthews, 1949) to any order in a perturbation theory for Lagrangians with arbitrary derivative coupling terms; these results were obtained without any reference to the Hamiltonian. Relation (3.31b) expresses the explicitly covariant S matrix as a T product once pure noncovariant contractions have been carried out.

In order to show that the covariant PDECC formalism properly connects the asymptotic free-field quantities, using $T^* = TT_n^{-1}$, we rewrite the right side of (3.12) as

$$S(\mathbf{g}) T^* F(x, y, \ldots) = T\{ S(\mathbf{g}) T^* F_{\mathrm{in}}(x, y, \ldots) + d(x, y, \ldots) \} \tag{3.32a}$$

$$d(x, y, \ldots) = [T_n^{-1}(S; F_{\mathrm{in}}) - 1] S(\mathbf{g})(T^* F_{\mathrm{in}}(x, y, \ldots)) \tag{3.32b}$$

$$d(x, y, \ldots)|_{\mathbf{g}=0} = 0 \tag{3.32c}$$

where we took into account relations (2.20) and the fact that $T_n^s T^* = T^*$, $T_n^s S = S$, $s = \pm 1$ [compare with (2.24)]. We notice that as a result of pure noncovariant contractions [compare with (2.3) and Appendix A] between the S matrix (expressed as a T^* product) and $T^* F_{\text{in}}(x, y, \ldots)$, the quantity $d(x, y, \ldots)$ (unless identically equal to zero) generally contains adiabatic factors $e(x)$, $e(y)$, ... Of course, these adiabatic factors will switch off the coupling constants in the infinite past and future ($\mathbf{g} \to 0$), which, in view of (3.32c), implies

$$d(x, y, \ldots) \to 0 \qquad \text{as} \quad t_x, t_y, \ldots \to \mp\infty \tag{3.32d}$$

where in these limits the difference between any two times is finite. Now expressing the S matrix as a T product [relation (3.31b)], from (3.32) we obtain the identity $ST^* F_{\text{in}}(x, y, \ldots) = ST^* F_{\text{in}}(x, y, \ldots)$ at $t_x, t_y, \ldots = -\infty$ and the relation (3.5) at $t_x, t_y, \ldots = +\infty$. Asymptotic relations (3.3) and (3.4) are simply consequences of (3.5) [see the discussion after (3.5)]. This shows that the covariant PDECC formalism relates properly all asymptotic free-field quantities.

Let us illustrate the properties of d from relations (3.22) on a model of a neutral scalar field $\sigma(x)$ with mass m interacting with symmetric tensor $j_{\mu\nu}(x)$ depending on fields different than $\sigma(x)$:

$$\mathscr{L}_0(x) = -\tfrac{1}{2}[(\partial_\mu \sigma(x))(\partial^\mu \sigma(x)) + m^2 \sigma^2(x)] \tag{3.33a}$$

$$\mathscr{L}_{\text{int}}(x) = e(x) g j_{\mu\nu}(x)\, \partial^\mu \partial^\nu \sigma(x) \tag{3.33b}$$

where we introduced explicitly the adiabatic factor $e(x)$. Here we wish to study the asymptotic properties of the energy-momentum tensor for just the σ field. We start with $T_{\text{in}}^{\mu\nu}(x)$, determined from $\mathscr{L}_0^{\text{in}}(x)$ to be

$$\begin{aligned} &T_{\text{in}}^{\mu\nu}(x) = :t_{\text{in}}^{\mu\nu}(x): \\ &t_{\text{in}}^{\mu\nu}(x) = \tfrac{1}{2}[(\partial^\mu \sigma_{\text{in}}(x))(\partial^\nu \sigma_{\text{in}}(x)) + (\partial^\nu \sigma_{\text{in}}(x))(\partial^\mu \sigma_{\text{in}}(x))] + g^{\mu\nu} \mathscr{L}_0^{\text{in}}(x) \\ &t_{\text{in}}^{\mu\nu}(x) = t_{\text{in}}^{\nu\mu}(x) \end{aligned} \tag{3.34a}$$

The interpolating $T^{\mu\nu}(x)$ is given as

$$S(g) T^{\mu\nu}(x) = T^* S(g) T_{\text{in}}^{\mu\nu}(x) = T\{S(g) T_{\text{in}}^{\mu\nu}(x) + d^{\mu\nu}(x)\} \tag{3.34b}$$

Now we wish to find whether

$$d^{\mu\nu}(x) = [T_n^{-1}(S; T_{\text{in}}^{\mu\nu}) - 1] S(g) T_{\text{in}}^{\mu\nu}(x) \tag{3.35a}$$

vanishes as $t_x \to \pm\infty$, where we take into account

$$T_n{}^s S = S,\ T_n{}^s T^{\mu\nu}_{\text{in}}(x) = T^{\mu\nu}_{\text{in}}(x),\ s = \pm 1$$

Evaluating (3.35a) to $O(g)$ by utilizing $[\partial_\rho \partial_\sigma \sigma_{\text{in}}(y)^{\cdot} \partial_\mu \sigma_{\text{in}}(x)^{\cdot}]''$ from Appendix A, after some work we obtain

$$\begin{aligned} d^{\mu\nu}(x) = ge(x)\{&[\partial_4 T^* j_{\rho\sigma}(x)][n^\rho n^\sigma (n^\mu \partial^\nu + n^\nu \partial^\mu) \\ &+ g^{\mu\nu} n^\rho n^\sigma \partial_4]\sigma_{\text{in}}(x) + [\partial^k T^* j_{\rho\sigma}(x)][n^\rho n^\sigma (g^\mu_k + g^\nu_k) \\ &+ (n^\mu + n^\nu)(n^\rho g^\sigma_k + n^\sigma g^\rho_k) + g^{\mu\nu}(n^\rho g^\sigma_k + n^\sigma g^\rho_k)\,\partial_4 \\ &- g^{\mu\nu} n^\rho n^\sigma \partial_k]\sigma_{\text{in}}(x)\} + O([ge(x)]^2) \end{aligned} \tag{3.35b}$$

where it is clear that adiabatic factor $e(x)$ appears also in higher order terms. Consequently, (3.35b) $\to 0$ as $t_x \to \mp\infty$; this gives from (3.34b) [compare with (3.3)]

$$S(g) T^{\mu\nu}_{\text{out}}(x) = T^{\mu\nu}_{\text{in}}(x) S(g) \tag{3.36}$$

where $T^{\mu\nu}_{\text{out}}$ is given by (3.34b) with σ_{in} replaced by σ_{out}. Of course, we can continue this study and, as an example of nonlocal observables, we could study the nonlocal currents (Sǒln, 1968). Again one would find that the adiabatic factors assure the correct relationships between in and out observables. Therefore, we can say that the explicitly covariant S matrix describes the entity of all possible results of measurements at the infinite future when the state of the infinite past has been specified. Let us point out that in our formalism an observable is a constant of motion only if $[O_{\text{in}}, S] = 0$. Here $T^{\mu\nu}$ is not a constant of motion, since it is the energy-momentum tensor for the σ field only.

Finally, we discuss the interpolating fields themselves. For a Lagrangian density with higher derivatives, one can write the generalized Euler–Lagrange equation

$$\frac{\partial \mathscr{L}(x)}{\partial \phi(x)} - \partial_\mu \frac{\partial \mathscr{L}(x)}{\partial \partial_\mu \phi(x)} + \partial_\mu \partial_\nu \frac{\partial \mathscr{L}(x)}{\partial \partial_\mu \partial_\nu \phi(x)} - \cdots = 0 \tag{3.37}$$

On the other hand, setting $F = \phi$, $G = \cdots = Q = 0$ in (3.25), we obtain

$$\frac{1}{i}\frac{\partial}{\partial g_i}\phi(x) = \int d^4y\, [T^*(\phi(x)\mathscr{L}_i^*(y)) - \mathscr{L}_i^*(y)\phi(x)] \tag{3.38}$$

The solution of (3.38), of course, is given by relations (3.6). Are relations (3.6) also the solution of the Euler–Lagrange equation (3.37)? We believe so. From (3.9) and (3.19) (with $\lambda \to 1$), we can write

$$\phi(x) = S^{\dagger}(\lambda) T^{*} S(\lambda) \phi_{\text{in}}(x)$$
$$= \phi_{\text{in}}(x) + i\lambda \int d^4y \, [T^{*}(\mathscr{L}^{\text{in}}_{\text{int}}(y)\phi_{\text{in}}(x)) - \phi_{\text{in}}(x)\mathscr{L}^{\text{in}}_{\text{int}}(y)]$$
$$+ O(\lambda^2) \tag{3.39}$$

Now applying (3.39) to the tensor derivative coupling from (3.33b), as an example (with $\lambda = 1$ and remembering that $j^{\mu\nu}$ depends on fields different than σ) we obtain

$$\sigma(x) = \sigma_{\text{in}}(x) + ig \int d^4y \, j^{\mu\nu}_{\text{in}}(y) \, \partial_\mu(y) \, \partial_\nu(y)$$
$$\times [T(\sigma_{\text{in}}(y)\sigma_{\text{in}}(x)) - \sigma_{\text{in}}(y)\sigma_{\text{in}}(x)] + O(g^2) \tag{3.40a}$$
$$= \sigma_{\text{in}}(x) + g \int d^4y \, \Delta_R(x-y)(\partial_\mu \partial_\nu j^{\mu\nu}_{\text{in}}(y)) + O(g^2) \tag{3.40b}$$

where derivatives were pulled to the left of the T^* product [compare with (2.1)] and two partial integrations were carried out. [Utilizing (3.40b), one easily verifies that indeed $\sigma^2(x) = S^{\dagger}(g) T^{*} S(g) \sigma^2_{\text{in}}(x)$, as expected from relations (3.14)]. Next from (3.37) we have exactly

$$\sigma(x) = \sigma_{\text{in}}(x) + g \int d^4y \, \Delta_R(x-y) \, \partial_\mu \partial_\nu j^{\mu\nu}(y) \tag{3.41}$$

Now if in (3.41) one further solves Euler–Lagrange equations for fields on which $j^{\mu\nu}$ depends, one gets $j^{\mu\nu}_{\text{in}}$ plus $O(g^2)$ terms, which we believe are the same as the $O(g^2)$ terms in (3.40b). Consequently, we conjecture that the solution for $\phi(x)$ as given by (3.39) is consistent with generalized the Euler–Lagrange equations (3.37). In any case, we can always accept (3.38) with its general solution (3.6a) as a covariant definition of the interpolating field $\phi(x)$.

4. GENERALIZED HAMILTONIAN DENSITY AND A FEW EXAMPLES

Relation (3.31b) suggests writing the S matrix in the Dyson form

$$S(\lambda) = T \exp\left[-i \int d^4x \, \mathscr{H}^{\text{in}}_{\text{int}}(x)\right] \tag{4.1}$$

where

$$\int d^4x\, \mathscr{H}^{\text{in}}_{\text{int}}(x) = i \ln\left\{T_n^{-1} \exp\left[i\lambda \int d^4x\, \mathscr{L}^{\text{in}}_{\text{int}}(x)\right]\right\} \tag{4.2}$$

where the limit $\lambda \to 1$ is understood after having done pertinent calculations. Relation (4.1) simply defines the generalized interaction part of the Hamiltonian density in terms of ϕ_{in}, $\partial_\mu\phi_{\text{in}}$, $\partial_\mu\partial_\nu\phi_{\text{in}}$, etc., for any $\mathscr{L}_{\text{int}}$.

When $\mathscr{L}_{\text{int}}$ is in usual form (3.21a), then taking into account (3.21b) (see also Appendix C), we have

$$\frac{d}{d\lambda}\mathscr{H}^{\text{in}}_{\text{int}}(x) = -\mathscr{L}_{\text{int}}(\phi_{\text{in}}(x), (\partial_\mu\phi(x))_{\text{in}}\,; \mathbf{g}) \tag{4.3a}$$

$$\exp\left\{i\int_0^\lambda d\lambda \int d^4x\, \mathscr{L}_{\text{int}}(\phi_{\text{in}}(x), (\partial_\mu\phi(x))_{\text{in}}\,; \mathbf{g})\right\}$$

$$= T_n^{-1} \exp\left\{i\lambda \int d^4x\, \mathscr{L}_{\text{int}}(\phi_{\text{in}}(x), \partial_\mu\phi_{\text{in}}(x); \mathbf{g})\right\} \tag{4.3b}$$

We see that relation (4.3b) is an identity at $\lambda=0$. If we apply $d/d\lambda$ on both sides of (4.3b) and evaluate the result at $\lambda=0$, we obtain relation (3.23). In general the left side is known from the PDECC and the canonical formalism (as we shall show shortly in the example of a chiral invariant Lagrangian). The action of T_n^{-1} on the right side of (4.3b) has to make it equal to the left side. Generally, this equality is impossible without $\delta_4(0)=0$. [In special cases such as the chiral-invariant Lagrangian (Sŏln, 1973; Gerstein *et al.*, 1971), one is able to add to $\mathscr{L}_{\text{int}}$ counter terms containing $\delta_4(0)$ which, in turn, cancel contributions with $\delta_4(0)$ from the action of T_n^{-1}.] Namely, $\delta_4(0)$ terms do not arise only from the local pure noncovariant contractions {e.g., from $[\partial_\mu\sigma_{\text{in}}(x)^{\bullet}\, \partial_\nu\sigma_{\text{in}}(x)^{\bullet}]^n$}, but also from the nonlocal ones when the integrals cannot "eat" all the delta functions, as in this example:

$$\int d^4x\, d^4y \cdots \delta_4(x-y)\, \delta_4(y-x) = \delta_4(0) \int d^4x \cdots$$

Since a term like this is clearly not present on the left side of (4.3b), it has to vanish, which, however, was already demanded independently by relation (3.23). In other words, $\delta_4(0)=0$ is sufficient to verify the whole of relation (4.3b).

Now we go to some specific examples. The first example is the case of the chiral-invariant Lagrangian density of massless pions (Sŏln, 1973;

Gerstein *et al.*, 1971)

$$\begin{aligned}\mathscr{L}(x) &= -\tfrac{1}{2}\phi_{\mu,a}(x)G_{ab}(\phi(x))\phi_b{}^{\mu}(x)\\ \phi_{\mu,a}(x) &= \partial_\mu\phi_a(x)\end{aligned} \tag{4.4a}$$

where $\phi_a(x)$ denotes the pion field. The dependence of G_{ab} on the pion field is determined by the requirement that $\mathscr{L}$ be $SU(2)\times SU(2)$ invariant (Weinberg, 1968); different $SU(2)\times SU(2)$ nonlinear-representation assignments for ϕ_a require different G's (Gerstein *et al.*, 1971; Weinberg, 1968). We write (Sŏln, 1973)

$$G_{ab}(x) = \delta_{ab} - \lambda\bar{G}_{ab}(\phi(x)) \tag{4.4b}$$

$$\lambda\mathscr{L}_{\text{int}}(x) = \frac{\lambda}{2}\,\phi_{\mu,a}(x)\bar{G}_{ab}(\phi(x))\phi^{\mu,b}(x) \tag{4.4c}$$

where the physical coupling constant is absorbed in $\bar{G}$. We now evaluate pure noncovariant contractions in (4.2),

$$\begin{aligned}\int \mathscr{H}_{\text{int}}^{\text{in}}(x)\,d^4x = i\ln\Big\{1 + i\lambda\int d^4x\,\mathscr{L}_{\text{int}}^{\text{in}}(x)\\ &+\frac{\lambda^2}{2}\int d^4x\,d^4y\,[-\mathscr{L}_{\text{int}}^{\text{in}}(x)\mathscr{L}_{\text{int}}^{\text{in}}(y)\\ &+\tfrac{1}{2}\delta_4(0)\delta_4(x-y)(n^2)^2\,\text{Tr}\,\bar{G}^2(\phi_{\text{in}}(x))\\ &-i\delta_4(x-y)n^\mu n^\nu\phi_{\mu,a}^{\text{in}}(x)\bar{G}_{ab}^2(\phi_{\text{in}}(x))\phi_{\nu,b}^{\text{in}}(x)] + O(\lambda^3)\Big\}\\ &= \int d^4x\Big\{-\lambda\mathscr{L}_{\text{int}}^{\text{in}}(x)\\ &+\frac{\lambda^2}{2}n^\mu n^\nu\phi_{\mu,a}^{\text{in}}(x)\bar{G}_{ab}^2(\phi_{\text{in}}(x))\phi_{\nu,b}^{\text{in}}(x) + O(\lambda^3)\Big\}\end{aligned}$$

Here, consistent with relations (2.36a), $\delta_4(0)$ is set to zero. The nth term in this series is easily deduced to be

$$\frac{\lambda^n}{2}n^\mu n^\nu\phi_{\mu,a}^{\text{in}}(x)\bar{G}_{ab}^n(\phi_{\text{in}}(x))\phi_{\nu,b}^{\text{in}}(x)$$

so that the result is

$$\mathscr{H}^{\text{in}}_{\text{int}}(x) = -\lambda \mathscr{L}^{\text{in}}_{\text{int}}(x) + \frac{n^\mu n^\nu}{2}\, \phi^{\text{in}}_{\mu,a}(x) \times \left[\frac{\lambda^2 \bar{G}^2(\phi_{\text{in}}(x))}{1-\lambda\bar{G}(\phi_{\text{in}}(x))}\right]_{ab} \phi^{\text{in}}_{\nu,b}(x) \tag{4.5}$$

Result (4.5) has to be consistent with (4.3a), which follows from the canonical formalism. To prove this, consistent with (4.4a), we start with the canonical momentum π_a conjugate to ϕ_a,

$$\pi_a(x) = [\delta_{ab} - \lambda \bar{G}_{ab}(\phi(x))]\phi_{4,b}(x) \tag{4.6}$$

Since

$$\pi^{\text{in}}_a(x) = \phi^{\text{in}}_{4,a}(x) = [\delta_{ab} - \lambda \bar{G}_{ab}(\phi_{\text{in}}(x))][\phi_{4,b}(x)]^{\text{in}}$$
$$\phi^{\text{in}}_{r,a}(x) = [\phi_{r,a}(x)]^{\text{in}}$$

we then have in general

$$[\phi_{\mu,a}(x)]^{\text{in}} = \left\{g_{\mu\nu} - n_\mu n_\nu\left[\frac{1}{1-\lambda\bar{G}(\phi_{\text{in}}(x))} - 1\right]\right\}_{a,b} \phi^{\nu,b}_{\text{in}}(x) \tag{4.7}$$

By taking into account that $\bar{G} = \bar{G}^T$ (T stands for "transpose" in unitary space), direct evaluation gives

$$\begin{aligned}
&\mathscr{L}_{\text{int}}(\phi^a_{\text{in}}(x), [\phi^{\mu,a}(x)]_{\text{in}}; g) \\
&= -\tfrac{1}{2}[\phi_{\mu,a}(x)]^{\text{in}} \bar{G}_{ab}(\phi_{\text{in}}(x))[\phi^{\mu,b}(x)]_{\text{in}} \\
&= -\tfrac{1}{2}\phi^{\text{in}}_{\mu,a}(x)\left\{\left[g^{\mu\rho} - n^\mu n^\rho\left(\frac{1}{1-y} - 1\right)\right]\right. \\
&\quad \times \bar{G}\left.\left[g_\rho{}^\nu - n_\rho n^\nu\left(\frac{1}{1-y} - 1\right)\right]\right\}_{ab} \phi^{\text{in}}_{\nu,b}(x) \\
&= -\tfrac{1}{2}\phi^{\text{in}}_{\mu,a}(x)\left\{\bar{G}\left[g^{\mu\nu} + n^\mu n^\nu\left(1 - \frac{1}{(1-y)^2}\right)\right]\right\}_{ab} \phi^{\text{in}}_{\nu,b}(x)
\end{aligned} \tag{4.8}$$

where $y = \lambda\bar{G}(\phi_{\text{in}}(x))$ and $\bar{G} = \bar{G}(\phi_{\text{in}}(x))$. Differentiating relation (4.5) with respect to λ, one obtains at once the negative of relation (4.8).

The S matrix for the chiral-invariant Lagrangian density [relations (4.4)] becomes explicitly

$$S(\lambda)=T^* \exp\left\{i \int d^4x \frac{\lambda}{2} \phi^{\text{in}}_{\mu,a}(x)\bar{G}_{ab}(\phi_{\text{in}}(x))\phi^{\mu,b}_{\text{in}}(x)\right\} \tag{4.9}$$

This expression does not have the counterterm with $\delta_4(0)$ (now equal to zero) which was introduced earlier (Sŏln, 1973; Gerstein *et al.*, 1971) to explicitly eliminate the worst divergences in the perturbation theory which violate the Adler condition for π–π scattering. However, it is again the dimensional regularization that takes care of the Adler condition. For example, it can be shown (Tataru, 1975) that all one-loop diagrams in the soft-pion limit (i.e., when all external momenta are zero) contain $\delta_4(0)$ and therefore are vanishing in the framework of dimensional regularization; therefore, one can say that the Adler condition is automatically satisfied in the one-loop approximation with this regularization. We believe that relation (4.9) provides the proper expression to all orders of the perturbation theory, providing that dimensional regularizations are employed for diagrams.

The previous example, although quite complicated, allowed us to define the Hamiltonian density canonically in the usual manner. As such, it served as a verification for the expression of the generalized Hamiltonian density from the covariant PDECC formalism. When $\mathscr{L}_{\text{int}}$ contains higher derivatives, as mentioned, it is possible to remove the second and higher time derivatives from $\mathscr{L}$ by carrying out either covariant or noncovariant field transformations (Barua and Gupta, 1977). However, as a rule the resulting Hamiltonian density is seldom in a simple form. As a consequence, the Dyson form of the S matrix is also very complicated. In fact, we can see this also within the covariant PDECC formalism in the example of the derivative tensors coupling from (3.33b). We shall assume for simplicity that $j^{\mu\nu}$, which depends on fields different than σ, also does not depend on field derivatives. Then applying relation (4.2) (at $\lambda=1$) to $\mathscr{L}_{\text{int}}$ from (3.33b), we obtain

$$\begin{aligned}\mathscr{H}^{\text{in}}_{\text{int}}(x) &= -\mathscr{L}^{\text{in}}_{\text{int}}(x)-\frac{i}{2}(T_n^{-1}-1)\int d^4y\, \mathscr{L}^{\text{in}}_{\text{int}}(y)\mathscr{L}^{\text{in}}_{\text{int}}(x)+O(g^3)\\ &= -\mathscr{L}^{\text{in}}_{\text{int}}(x)-g^2 j^{\mu\nu}_{\text{in}}(x)\{[n_\mu n_\nu g_\alpha{}^k g_\beta{}^l+2n_\mu n_\alpha g_\nu{}^k g_\beta{}^l]\,\partial_k\partial_l\\ &\quad +2n_\mu n_\nu n_\alpha g_\beta{}^k\partial_k\partial_4+\tfrac{1}{2}n_\mu n_\nu n_\alpha n_\beta(\partial_4^2-\nabla^2-m^2)\}j^{\alpha\beta}_{\text{in}}(x)\\ &\quad +O(g^3)\end{aligned} \tag{4.10}$$

where the pure noncovariant contractors from Appendix A were taken into account. Clearly, even to $O(g^2)$, $\mathscr{H}^{\text{in}}_{\text{int}}$ is rather complicated. The S matrix in

Dyson form with this $\mathscr{H}_{\rm int}^{\rm in}$ [valid only to $O(g^2)$] becomes also rather complicated. This should be contrasted with the equivalent covariant S matrix,

$$S(g) = T^* \exp\left[ig \int d^4x\, j_{\rm in}^{\mu\nu}(x)\, \partial_\mu \partial_\nu \sigma_{\rm in}(x)\right] \tag{4.11}$$

which is straightforwardly simple, unitary, and valid to all orders in g.

5. DISCUSSION AND CONCLUSION

By employing the covariant PDECC formalism, we have developed the covariant perturbation theory for the S matrix and the interpolating fields when the coupling terms in the Lagrangian density involve arbitrary (first and higher) derivatives. The remarkable thing is, however, that this theory was formulated directly in terms of the Lagrangian density without any reference to a Hamiltonian density, which, for a higher derivative Lagrangian, is generally very difficult to obtain (Barua and Gupta, 1977). The advantage of this approach is evident if one considers the symmetry properties of the system, which are usually expressed through Lagrangians.

Of course, it was the introducion of the (purely) noncovariant T_n^s ($s = \pm 1$) products that allowed us to avoid the Hamiltonian density altogether. In fact, we have turned things around and, as shown in Section 4, with the help of the T_n^{-1} product defined the generalized Hamiltonian density. Furthermore, it is T_n^{-1} that brings T to the left in T^* ($T^* = TT_n^{-1}$), and as such allows us to interpret T^* as a covariant time-ordering operator. This, in turn, facilitates definitions of interpolating field quantities and shows that the explicitly covariant S matrix correctly relates in and out observables.

A rather gratifying fact is that, from the requirement that the S matrix from a canonical formalism coincides with our explicitly covariant S matrix for Lagrangian densities with first derivatives, we obtain $\delta_4(0) = 0$ [and, through the dimensional regularization, also $(\partial_{\mu_1}\partial_{\mu_2} \cdots \delta_4)(0) = 0$]. That $\delta_4(0)$ should be set to zero (within the dimensional regularization) has been found already in Bernard and Duncan (1975) and Barua and Gupta (1977). There for Lagrangian densities with higher derivatives, it is possible to construct covariant S matrices with the help of Hamiltonian densities if $\delta_4(0) = 0$. However, in these cases one has a chance to demonstrate the validity of Matthews' (1949) theorem mostly to low orders in the perturbation theory (Bernard and Duncan, 1975; Barua and Gupta, 1977). In contrast, explicitly covariant expressions (3.28) and (3.31a) for the S matrix make Matthews' theorem correct to all orders of the perturbation theory.

Let us point out that even for the case of (second) derivative tensor coupling (3.33b), the calculated $\mathscr{H}_{\text{int}}^{\text{in}}$, although complicated, is still a local operator, as can be seen from (4.10). Consequently, even in the original Dyson form, (4.1), the S matrix is conventionally unitary, as opposed to nonlocal interactions, where the S may become nonunitary (Hata, 1989).

APPENDIX A

In this Appendix we give a few examples of pure noncovariant contractions. In these examples, the free field is a scalar (or pseudoscalar) field $\phi_f^a(x)$, $f=$ in, out, and a is some internal index. Consistent with (2.4) and (2.8a) for two free multiderivative fields, the T product can be written as

$$T\,1_f 2_f = T^* e^{\hat{N}_f} 1_f 2_f = T^* 1_f 2_f + \hat{N}_f 1_f 2_f \tag{A1}$$

where, because $\hat{N}_f 1_f 2_f$ is a c-number, $T^* \hat{N}_f 1_f 2_f = \hat{N}_f 1_f 2_f$. As a consequence of (A1), the pure noncovariant contraction of two free multiderivative fields is numerically given as

$$\begin{aligned}
&\hat{N}_f(\partial_{\mu_1}\partial_{\mu_2}\cdots\phi_f^a(x))(\partial_{\nu_1}\partial_{\nu_2}\cdots\phi_f^b(y))\\
&\quad\equiv[\partial_{\mu_1}\partial_{\mu_2}\cdots\phi_f^a(x)^{\bullet}\partial_{\nu_1}\partial_{\nu_2}\cdots\phi_f^b(y)^{\bullet}]^n\\
&\quad=\langle 0;f|(T-T^*)(\partial_{\mu_1}\partial_{\mu_2}\cdots\phi_f^a(x))\\
&\qquad\times(\partial_{\nu_1}\partial_{\nu_2}\cdots\phi_f^b(y))|0;f\rangle
\end{aligned} \tag{A2}$$

Relation (A2) yields the following pure noncovariant contractions [compare also with Barua and Gupta (1977) and Šoln (1973)]

$$[\phi_f^a(x)^{\bullet}\phi_f^b(y)^{\bullet}]^n=0 \tag{A3}$$

$$[\partial_\mu\phi_f^a(x)^{\bullet}\phi_f^b(y)^{\bullet}]^n=0 \tag{A4}$$

$$[\partial_\mu\phi_f^a(x)^{\bullet}\,\partial_\nu\phi_f^b(y)^{\bullet}]^n=-in_\mu n_\nu\,\delta_{ab}\,\delta_4(x-y) \tag{A5}$$

$$[\partial_\mu\partial_\nu\phi_f^a(x)^{\bullet}\phi_f^b(y)^{\bullet}]^n=in_\mu n_\nu\,\delta_{ab}\,\delta_4(x-y) \tag{A6}$$

$$\begin{aligned}
&[\partial_\mu\partial_\nu\phi_f^a(x)^{\bullet}\,\partial_\rho\phi_f^b(y)^{\bullet}]^n\\
&\quad=-i\delta_{ab}\{n_\mu n_\nu n_\rho\,\partial_4(x)\,\delta_4(x-y)\\
&\qquad+(n_\mu n_\nu g_\rho{}^i+n_\mu n_\rho g_\nu{}^i+n_\nu n_\rho g_\mu{}^i)\,\partial_i(x)\,\delta_4(x-y)\}
\end{aligned} \tag{A7}$$

$$
\begin{aligned}
&[\partial_\mu \partial_\nu \phi_f^a(x)^\bullet\, \partial_\rho \partial_\sigma \phi_f^b(y)^\bullet]^n \\
&\quad = i\delta_{ab}\{(n_\rho n_\sigma g_\nu{}^i g_\mu{}^j + n_\nu n_\sigma g_\rho{}^i g_\mu{}^j \\
&\qquad + n_\nu n_\rho g_\sigma{}^i g_\mu{}^j + n_\mu n_\nu g_\rho{}^i g_\sigma{}^j \\
&\qquad + n_\mu n_\rho g_\nu{}^i g_\sigma{}^j + n_\mu n_\sigma g_\nu{}^i g_\rho{}^j)\, \partial_i(x)\partial_j(x)\delta_4(x-y) \\
&\qquad + (n_\rho n_\sigma n_\nu g_\mu{}^i + n_\nu n_\sigma n_\mu g_\rho{}^i \\
&\qquad + n_\nu n_\rho n_\mu g_\sigma{}^i + n_\rho n_\sigma n_\mu g_\nu{}^i)\, \partial_i(x)\partial_4(x)\delta_4(x-y) \\
&\qquad + n_\mu n_\nu n_\rho n_\sigma[\partial_4^2(x) + \partial_i(x)\,\partial^i(x) - m^2]\delta_4(x-y), \quad \text{etc.}
\end{aligned}
\tag{A8}
$$

where m is the mass associated with the field $\phi_f^a(x)$. As one sees, although straightforward to evaluate, the complexity of pure noncovariant contractions increases with the number of derivatives acting on the free fields. The exponential representation of the T_n product helped to give a precise definition of pure noncovariant contractions. Actually, N_f even can be given explicitly in terms of bilinear functional derivatives with respect to multiderivative free fields, which, however, due to the lack of space, were not used here (Šoln, 1990).

APPENDIX B

In the scheme of dimensional regularization, the integrals in the momentum space are evaluated in n dimensions rather than four dimensions. Here, the n-dimensional "Minkowski" space has one timelike and $n-1$ spacelike dimensions ('t Hooft and Veltman, 1972). From 't Hooft and Veltman (1972) we have the following equations for arbitrary n:

$$
\begin{aligned}
&\int d^n p\, \frac{1}{(p^2+2kp+m^2)^\alpha} = \frac{i\pi^{n/2}(m^2-k^2)^{n/2-\alpha}\Gamma(\alpha-n/2)}{\Gamma(\alpha)} \\
&\quad \Rightarrow \int d^n p\, \frac{1}{(p^2)^\alpha} = 0, \qquad n/2 > \alpha \\
&\quad \Rightarrow \int d^n p = 0
\end{aligned}
\tag{B1}
$$

$$
\begin{aligned}
&\int d^n p\, \frac{p_\mu}{(p^2+2kp+m^2)^\alpha} = \frac{i\pi^{n/2}(m^2-k^2)^{n/2-\alpha}\Gamma(\alpha-n/2)}{\Gamma(\alpha)}(-k^\mu) \\
&\quad \Rightarrow \int d^n p\, \frac{p_\mu}{(p^2)^\alpha} = 0, \qquad \frac{n}{2} > \alpha \\
&\quad \Rightarrow \int d^n p\, p_\mu = 0
\end{aligned}
\tag{B2}
$$

Following 't Hooft and Veltman (1972) by differentiating the first relation in (B1) arbitrarily many times with respect to k, we end up with

$$\int d^n p \frac{p_\mu p_\nu p_\lambda \cdots p_\rho}{(p^2+2kp+m^2)^\alpha} = \frac{i\pi^{n/2}(m^2-k^2)^{n/2-\alpha}}{\Gamma(\alpha)} t_{\mu\nu\lambda\cdots\rho} \tag{B3}$$

where the tensor $t_{\mu\nu\lambda\cdots\rho}$ is made up of vectors $k_\mu, k_\nu, k_\lambda, \ldots, k_\rho$, and of metric tensors $g_{\mu\nu}, g_{\mu\lambda}, g_{\nu\lambda}, \ldots$, and, of course, depends on m^2 and k^2. From dimensional arguments we see that the dependence of $t_{\mu\nu\lambda\cdots}$ on m^2 and k^2 is such that $t_{\mu\nu\lambda\cdots}$ is well behaved when $m^2 \to 0$ and $k^2 \to 0$ (where, while taking the limits $k^\mu \to 0$ and $m^2 \to 0$, we keep the ratio k^2/m^2 fixed). Consequently, using procedures similar to the ones from (B1) and (B2), we conclude that

$$\int d^n p\, p_\mu p_\nu p_\lambda \cdots p_\rho = 0 \tag{B4}$$

Relations (B1), (B2), and (B4) imply $\delta_4(0)=0$ and $(\partial_\mu \partial_\nu \partial_\lambda \cdots \partial_\rho \delta_4)(0) = 0$, which are the same as relations (2.36).

APPENDIX C

To explain relations (4.3a) and (3.21) within the framework of the canonical formalism, following Šoln (1973), we define the prime partial derivative with respect to the coupling constant g_i, $\partial'/\partial g_i$, with the properties

$$\frac{\partial'}{\partial g_i}\phi(x)=0, \qquad \frac{\partial'}{\partial g_i}\pi(x)=0 \tag{C1}$$

where π is the canonical momentum conjugate to ϕ. By taking into account that π can be expressed in terms of ϕ, $\partial_r\phi$, and $\dot{\phi}$ and as such may explicitly depend on g_i, taking into account (3.18), we have

$$\frac{\partial^*}{\partial g_i} = \frac{\partial'}{\partial g_i} + \sum_\phi \frac{\partial^* \pi}{\partial g_i}\frac{\partial}{\partial \pi} \tag{C2}$$

With $\partial\mathscr{H}/\partial\pi = -\dot{\phi}$, we then have

$$\frac{\partial^*\mathscr{H}}{\partial g_i} = \frac{\partial'\mathscr{H}}{\partial g_i} - \sum_\phi \dot{\phi}\frac{\partial^*\pi}{\partial g_i} \tag{C3}$$

which, when combined with $\mathscr{L} = \sum_\phi \pi\dot{\phi} - \mathscr{H}$, yields (Šoln, 1973)

$$\frac{\partial'\mathscr{H}}{\partial g_i} = -\frac{\partial^*\mathscr{L}}{\partial g_i} \tag{C4}$$

By writing $\mathscr{H}=\mathscr{H}_0+\mathscr{H}_{\text{int}}$ and $\mathscr{L}=\mathscr{L}_0+\lambda\mathscr{L}_{\text{int}}$, where $\mathscr{H}_0$ is the free part of the Hamiltonian density, varying the mathematical coupling constant λ between 0 and 1 while freezing the physical coupling constants [compare with (3.19)], and taking into account that $d'\mathscr{H}_0/d\lambda=0$, from relation (4), with $g_i\equiv\lambda$, we obtain

$$\frac{d'\mathscr{H}_{\text{int}}(x)}{d\lambda}=\mathscr{L}_{\text{int}}(x) \tag{C5}$$

where

$$\mathscr{H}_{\text{int}}(x)=\mathscr{H}_{\text{int}}(\phi(x), \partial_r\phi(x), \pi(x); \lambda; \mathbf{g})$$

and

$$\mathscr{L}_{\text{int}}(x)=\mathscr{L}_{\text{int}}(\phi(x), \partial_\mu\phi(x); \mathbf{g})$$

Now taking the $t_x\to-\infty$ asymptotic limit of (C5), we have

$$\frac{d'\mathscr{H}^{\text{in}}_{\text{int}}(x)}{d\lambda}=-\mathscr{L}_{\text{int}}(\phi_{\text{in}}(x), (\partial_\mu\phi(x))_{\text{in}}; \mathbf{g}) \tag{C6}$$

where $\mathscr{H}^{\text{in}}_{\text{int}}(x)=\mathscr{H}_{\text{int}}(\phi_{\text{in}}(x), \partial_r\phi_{\text{in}}(x), \pi_{\text{in}}(x); \lambda; \mathbf{g})$ and we took into account that $d\pi_{\text{in}}/d\lambda=0$. Relation (C6), which is the same as (4.3a), explains (3.21b) if one differentiates (4.1) with respect to λ.

Next, to demonstrate (3.22), we take $\phi(x)$ to be a scalar field. Then from (3.19) we have $\pi=\dot\phi+\lambda\,\partial\mathscr{L}_{\text{int}}/\partial\dot\phi$, which, after we take the $t_x\to-\infty$ limit, yields

$$(\partial_\mu\phi(x))_{\text{in}}=\partial_\mu\phi_{\text{in}}(x)-\lambda n_\mu \times\frac{\partial\mathscr{L}_{\text{int}}(\phi_{\text{in}}(x), (\partial_\mu\phi(x))_{\text{in}}; \mathbf{g})}{\partial(\dot\phi(x))_{\text{in}}} \tag{C7}$$

where we took into account that $\pi_{\text{in}}=\dot\phi_{\text{in}}$. This gives

$$\mathscr{L}_{\text{int}}(\phi_{\text{in}}(x), (\partial_\mu\phi(x))_{\text{in}}; \mathbf{g})=\mathscr{L}_{\text{int}}(\phi_{\text{in}}(x), \partial_\mu\phi_{\text{in}}(x); \mathbf{g})+O(\lambda) \tag{C8}$$

which is the same as (3.22).

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